# Virasoro constraints, Vertex operator algebras, and Wall-crossing

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I. History of Virasoro constraints.

II. Descendent invariants & Virasoro operators

III. Joyce's vertex algebra

IV. Wall-crossing

## I. History of Virasoro constraints

**Virasoro algebra**

$$\mathrm{Vir} := \mathrm{span}_{\mathbb{C}}\left( \{L_n\}_{n\in\mathbb{Z}} \sqcup \{c\} \right)$$

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{\delta_{n+m,0}\cdot(n^3-n)}{12}\cdot c$$

$$\mathrm{Vir}_{\geq -1} := \mathrm{span}_{\mathbb{C}}\{L_n\}_{n\geq -1} \subset \mathrm{Vir} \quad \text{Lie subalgebra.}$$

**Witten's conjecture**

$$Z(\lambda, t_0, t_1, \cdots) := \exp\left( \sum_{g\geq 0} \lambda^{2g-2} \sum_{\substack{n\geq 0 \\ k_1,\cdots,k_n\geq 0}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n} \; \frac{t_{k_1}\cdots t_{k_n}}{n!} \right)$$

$$\in \mathbb{Q}[\![\lambda^{\pm}, t_0, t_1, \cdots]\!] \circlearrowleft \mathrm{Vir}_{\geq -1}$$

e.g. $L_{-1} = -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=1}^{\infty} t_{i+1}\frac{\partial}{\partial t_i} \longleftrightarrow$ string equation

<u>Thm</u>. (Kontsevich) $\forall k\geq -1$, $L_k\left(Z(\lambda, t_0, t_1, \cdots)\right) = 0$.

<u>Rmk</u>. This determines $Z(\lambda, t_0, \cdots)$ uniquely given some initial data.

**GW theory** **Sheaf theory**

$$\begin{tikzcd}
\{\overline{\mathcal{M}}_{g,n}\}_{g,n} \arrow[d, rightsquigarrow] & & M_S^{H\text{-ss}}(r,c_1,c_2) \quad [\mathrm{vB}] \\
 & & S^{[n]} \arrow[u, rightsquigarrow] \\
\{\overline{\mathcal{M}}^{\bullet}_{g,n}(X,\beta)\}_{g,n} \arrow[rr, leftrightarrow, "{[\mathrm{MNOP}]}", "\dim X = 3"'] & & P_{n,\beta}(X) \arrow[u, rightsquigarrow, "{X = S\times\mathbb{P}^1,\ \beta = n\cdot f}"']
\end{tikzcd}$$

[EHX] [MOOP], [M]

<u>proven cases</u> (sheaf side)

- $*$ stationary case for $P_{n,\beta}(X)$ for toric 3-fold [MOOP]
- $*$ $S^{[n]}$ for $b_1(S)=0$ [M]
- $*$ $\sim 1600$ cases of $M_S^{H\text{-ss}}(r,c_1,c_2)$ for toric $S$ [vB]

<u>Q</u>. How general is this?

<u>Conjecture</u>. (Bojko-L-Moreira)

$$\int_{[M_X^{H\text{-ss}}(v)]^{\mathrm{vir}}} L_{\mathrm{wt}_0}(D) = 0 \quad \text{for all} \quad D \in \mathbb{D}^X .$$

assuming this exists

Rmk. this not only generalizes the previous conjectures but also improve the presentation by avoiding
* careful choice of a universal sheaf
* something else than $\mathrm{Vir}_{\geq -1}$
* division by the rank

Thm. (BLM) Virasoro constraints hold for
* $M_C^{ss}(r,d)$
* $M_S^{H\text{-}ss}(r, c_1, c_2)$
* $M_S^{H\text{-}ss}(0, \beta, \chi)$ assuming the required wall-crossing formulas.

S has only $(p,p)$ classes.

Eg. $M = M_C^{ss}(2, \Lambda)$ ← odd degree

$c_2(\mathrm{End}\,\mathbb{F}) \rightsquigarrow$ Newstead classes $\alpha \in H^2(M)$, $\beta \in H^4(M)$, $\gamma \in H^6(M)$.

Virasoro constraints of $M$ $\Longleftrightarrow$ $(g-k)\int_M \alpha^n \beta^m \gamma^k = -2n \int_M \alpha^{n-1}\beta^{m-1}\gamma^{k+1}$.

↑ monodromy invariance

$\therefore$ $\int_M \alpha^n \beta^m \gamma^k$ is known.

## II. Descendent invariants & Virasoro operators

Suppose that $M = M_X^{H\text{-ss}}(v)$ satisfies

1) $\mathrm{Ext}^i(F,F) = 0 \quad \forall i \geq 3$
2) H-ss = H-st
3) $\exists$ universal sheaf $\mathbb{F}$ on $M \times X$.

$\rightsquigarrow [M]^{vir} \in H_{2\cdot \underset{\sim}{vd}}(M)$.

$1-\chi(F,F)$.

### Descendent invariants

$$H^*(M,\mathbb{C}) \otimes H_*(M,\mathbb{C}) \longrightarrow \mathbb{C}$$

$$\xi^{\mathbb{F}} \uparrow \mathbb{D}^X \qquad \rotatebox{90}{$\in$}\ [M]^{vir} \qquad \rotatebox{90}{$\in$}\ \int_{[M]^{vir}} \xi^{\mathbb{F}}(D).$$

A super-comm. ring homomorphism

image = taut. subring
kernel = taut. relations $\longleftarrow$ does not see $[M]^{vir}$

$$\mathbb{D}^X \xrightarrow{\ \xi^{\mathbb{F}}\ } H^*(M,\mathbb{C})$$

$$\mathrm{ch}_i(\gamma) \longmapsto \pi_{M*}\left(\mathrm{ch}_{i+\dim_{\mathbb{C}}X-p}(\mathbb{F}) \cdot \pi_X^*\gamma\right)$$

$i \geq 0,\ \gamma \in H^{p,*}(X,\mathbb{C})$ $\qquad\qquad$ $\in H^{i,*}(M,\mathbb{C})$

Q. What are the structures of $\displaystyle\int_{[M]^{vir}} \xi^{\mathbb{F}}(D)$ ?

e.g. dimension constraints, algebraicity constraints.

# Virasoro operators

$$\mathrm{Vir}_{\geq -1} = \mathrm{span}_{\mathbb{C}}\{L_n\}_{n\geq -1} \curvearrowright \mathbb{D}^X \quad \text{by} \quad L_n = R_n + T_n$$

* derivation s.t. $R_n(\mathrm{ch}_i(\gamma)) := i(i+1)\cdots(i+n)\,\mathrm{ch}_{i+n}(\gamma)$
* multiplication by $T_n := \sum_{\substack{a+b=n \\ a,b\geq 0}} a!\,b! \sum_s (-1)^{\dim_{\mathbb{C}} X - p_s^L} \underbrace{\mathrm{ch}_a(\gamma_s^L)\,\mathrm{ch}_b(\gamma_s^R)}$

$$\Delta_* \mathrm{td} X = \sum_s \gamma_s^L \otimes \gamma_s^R .$$

<u>Lemma</u>. $[L_n, L_m] = (m-n)L_{n+m}$.

<u>Def</u>. (BLM) $L_{wt_0} := \sum_{n\geq -1} \frac{(-1)^n}{(n+1)!} L_n \circ (L_{-1})^{n+1}$ where does it come from?

<u>Lemma</u>. 1) $L_{wt_0} : \mathbb{D}^X \longrightarrow \mathbb{D}^X_{wt_0} := \ker(L_{-1})$

2)
$$\begin{tikzcd}
\mathbb{D}^X \arrow[r, dashed, "\xi^{\mathbb{F}}"] & H^*(M) \\
\mathbb{D}^X_{wt_0} \arrow[u, phantom, "\cup"] \arrow[ur, "\xi"'] \arrow[r, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

independent on the choice of $\mathbb{F}$.

<u>Recall the conjecture</u>:
$$\begin{tikzcd}
\mathbb{D}^X \arrow[r, "L_{wt_0}"] \arrow[rrr, bend right, "0"'] & \mathbb{D}^X_{wt_0} \arrow[r, "\xi"] & H^*(M) \arrow[r, "\int_{[M]^{vir}}"] & \mathbb{C}
\end{tikzcd}$$

## III. Joyce's vertex algebra

$\mathcal{M}_X = \mathrm{Map}(X, \mathrm{Perf}_{\mathbb{C}})$ stack of all perfect complexes.

<u>Thm</u>. (Joyce) $V := H_*(\mathcal{M}_X, \mathbb{C})$ is naturally a vertex algebra.

$$\left( V,\ 1 \in V,\ T : V \to V,\ Y : V \otimes V \to V((z)) \right)$$

vacuum vector: $0 \in \mathcal{M}_X$

translation: $B\mathbb{G}_m \times \mathcal{M}_X \to \mathcal{M}_X$

$\oplus : \mathcal{M}_X \times \mathcal{M}_X \to \mathcal{M}_X$

$\Theta := R\mathcal{H}om_\pi(\mathcal{F}_2, \mathcal{F}_1) \oplus R\mathcal{H}om_\pi(\mathcal{F}_1, \mathcal{F}_2)^\vee$

$$Y(a,z)b := (-1)^{\chi(a^{top}, b^{top})} \oplus_* \left( e^{zT} \otimes 1 \left( \underbrace{\hat{c}_z(\Theta)}_{\prod_{i \in I}(z + x_i)} \cap a \boxtimes b \right) \right)$$

<u>Q</u>. Why consider this?

$M = M_X^{H\text{-}ss}(v)$ as before $\rightsquigarrow$

$$\begin{tikzcd} M \arrow[r, "\mathbb{F}"] \arrow[dr, "i"', "\circ"] & \mathcal{M}_X \arrow[d, "\mathbb{G}_m\text{-gerbe}"] \\ & \mathcal{M}_X^{rig} \end{tikzcd}$$

$$\rightsquigarrow i_*[M]^{vir} \in H_*(\mathcal{M}_X^{rig}) \simeq H_*(\mathcal{M}_X) / T(H_*(\mathcal{M}_X))$$

**Upshot** : Interesting constructions in VA leads to geometric analogue.

| **Algebraic** | | **Geometric** |
|---|---|---|
| Lattice VA w.r.t. $\left(K^{\bullet}_{top}(\mathcal{X}), \chi_{sym}\right)$ | $\xleftrightarrow{[G]}$ | Joyce's VA on $H_*(\mathcal{M}_\mathcal{X})$ |
| Borcherds' Lie algebra $\left(\check{V} := V/T(V), [,]\right)$ | $\xleftrightarrow{[J, GJT]}$ | wall-crossing between $[M^{H\text{-ss}}_{\mathcal{X}}(v)]^{vir} \in \check{H}_*(\mathcal{M}_\mathcal{X}, \mathbb{C})$. |
| conformal element $\omega$ & primary/physical states | $\xleftrightarrow{[BLM]}$ | Virasoro constraints for $M^{H\text{-ss}}_{\mathcal{X}}(v)$ |

**Def.** $\omega \in V$ is a conformal element if

* $Y(\omega, z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ defines $\mathrm{Vir} \circlearrowright V$
* $L_{-1} = T$
* $L_0$ diagonalizable

**Def.** 1) $P_\lambda := \{ a \in V \mid L_0(a) = \lambda \cdot a \,,\ L_k(a) = 0 \ \forall k \geq 1 \}$

2) $\check{P} := P_1 / T(P_0) \hookrightarrow \check{V}$ Lie subalgebra of primary states.

<u>Thm.</u> (BLM) $X$ : curve, surface with $p_g = 0$, or Fano 3-fold.

$\exists$ natural conformal element $\omega \in V^{\text{pair}}$

s.t. $M = M_X^{H\text{-}ss}(v)$ satisfies the Virasoro constraints iff

$[M]^{\text{vir}} \in \check{V}$ is a <u>primary state</u> w.r.t. $\omega$.

<u>Rmk.</u> 1) Primary states are very important in VOA theory

e.g. proof of monstrous moonshine

2) The fact that $\underline{\check{P}} \subset \check{V}$ forms a $\underline{\text{Lie}}$ subalgebra is important.

Virasoro constraints. wall-crossing

This follows from $\check{P} = \ker([-, \omega])$.

3) $[-, \omega] = \mathrm{Res}_z Y(-, z)\omega = [z^{-1}]\, e^{zL_{-1}} \circ \sum_{n \in \mathbb{Z}} L_n (-z)^{-n-2}$

$$= \sum_{n \geq -1} \frac{(-1)^n}{(n+1)!} (L_{-1})^{n+1} \circ L_n \longrightarrow \text{dual to } L_{wt_0} \curvearrowright \mathbb{D}^X.$$

IV. Wall-crossing

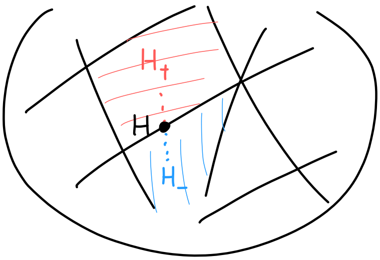
<u>Thm.</u> (Joyce) $X$ : curve or surface

"Wall-crossing formulas for $[M_X^{H\text{-}ss}(\alpha)]^{\text{inv}} \in (\check{V}, [,])$"

can be explicitly written in terms of $[,]$.

## Example (Simple wall-crossing)

Amp(X)




$$F \in M_X^{H_+\text{-ss}}(v) \setminus M_X^{H_-\text{-ss}}(v)$$

iff $0 \to F_1 \to F \to F_2 \to 0$, $F_i \in M_X^{H\text{-ss}}(v_i)$ ← irreducible

$*$ similarly for other complement with $F_1, F_2$ swapped

$$\Rightarrow \left[M_X^{H_+\text{-ss}}(v)\right]^{vir} - \left[M_X^{H_-\text{-ss}}(v)\right]^{vir} = \left[\left[M_X^{H\text{-ss}}(v_1)\right]^{vir}, \left[M_X^{H\text{-ss}}(v_2)\right]^{vir}\right]$$

key ideas in the proof (e.g. $S$ surface)

$$\begin{array}{ccc} P_S^{0+}(v) & \overset{t \in (0,\infty)}{\longleftarrow\!\sim\!\sim\!\sim} & P_S^{\infty}(v) = \begin{cases} \phi & rk(v) > 1 \\ \underbrace{S_\beta^{[0,m]}}_{\text{initial input } [M]} & rk(v) = 0 \end{cases} \\ \downarrow \pi & & \\ M_S^{H\text{-ss}}(v) & & \end{array}$$

$$\underbrace{\pi_*\left( c_{top}(T_\pi) \cap \left[P_S^{0+}(v)\right]^{vir} \right)}_{\text{projective bundle compatibility}} = \chi(v) \cdot \left[M_S^{H\text{-ss}}(v)\right]^{inv} + \underbrace{(\text{lower rank})}_{\text{rank induction}}$$

$\underline{Q}$. What about $M$ = moduli of objects in $D^b(X)$ ?

e.g. $P_{n,\beta}(X)$ parametrizes $I^\bullet = [\mathcal{O}_X \to F] \in D^b(X)$.

e.g. $X$ cubic 3-fold, $D^b(X) = \langle Ku(X), \mathcal{O}, \mathcal{O}(H) \rangle$

- $K_{num}(Ku(\mathbb{Z})) \simeq \mathbb{Z}^2$
- "unique" Serre invariant stability condition $\sigma \in Stab(Ku(X))$

$\rightsquigarrow \underbrace{M^{\sigma\text{-ss}}_{Ku(X)}(v)}$ projective variety smooth at stable points.

Virasoro constraints for these moduli spaces?

$\times$ Note that two example coincide at special case:

$$\underline{\underline{P_{1,\ell}(X)}} \simeq \underbrace{Fano(X)} \simeq M^{\sigma\text{-ss}}_{Ku(X)}((1,0))$$

Fano surface of lines in $X$.

Virasoro is proven by [Moreira] using Hodge theory.